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of rotation and the distances from the plate to I, S, and the disk D. Plotting wave radii as ordinates and the corresponding time intervals as abscissas, the tangent to the curve at any point gives the instantaneous velocity of sound at that distance from the source. The accompanying curve shows that the velocity varies from about 660 meters per sec. at a distance of 3.2 mm. from the sound source to 380 meters per second at a distance of 1.8 cm. Results have been obtained since the above curve was plotted for points much closer to the source and for distances up to a half meter.

The writer designates as "weak sparks" those produced when each condenser, L, consists of two Leyden jars; as "strong sparks," seven jars. The curve appears to show but little difference in the velocity of the resulting sound waves. However, considering the shortness of the sound pulses produced by such electric sparks, all are really intense waves at points near the source.

This investigation, made under a grant from the American Association for the Advancement of Science, will be published in full in the *Physical Review.*

CONFORMAL MAPPING OF A FAMILY OF REAL CONICS ON ANOTHER

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Note IV On Conformal Mapping Under Aid of Grant No. 207 From the Bache Fund

Let $z = x + yi$ and $w = u + vi$ be two complex variables, and let the analytic function $w = w(z)$ define a conformal map of the z-plane upon the w-plane. It is the purpose of the present note to determine all functions $w(z)$ such that there exists a family (containing at least one real parameter) of real conics in the z-plane which is mapped upon a family (obviously with an equal number of parameters) of real conics in the w-plane. The particular case when the conics in the w-plane are straight lines parallel to the real axis has been investigated by Von der Mühll[1] and Meyer.[2]

It is convenient to use the isometric coördinate $z = x + yi$ and $\bar{z} = x - yi$, and we begin by establishing the following general results:

When $\bar{z} = f(z, t)$, where f is analytic in both its arguments and t is a real parameter, represents a family of real curves in the z-plane, and similarly $\bar{w} = F(w, t)$ is a family of real curves in the w-plane, the necessary and sufficient condition that $w = w(z)$ shall map these two families upon each other is that the relation

$$\frac{\dfrac{\partial F}{\partial w}}{\dfrac{\partial F}{\partial t}}\frac{dw}{dz} = \frac{\dfrac{\partial f}{\partial z}}{\dfrac{\partial f}{\partial t}}$$

shall become an identity in z and t upon making $w = w(z)$.

In order that the z- and w-planes may be mapped upon an auxiliary Z-plane in such a manner that our two families of curves are mapped into the parallels to the real Z-axis, it is necessary and sufficient that the right-hand member in the relation above equals the product of a function of z by a function of t. When this is not the case, and our two families of curves are both algebraic, then $w(z)$ is necessarily an algebraic function of z.

Applying these general propositions to the various kinds of real conics, we obtain all the conformal maps having the required property, and replacing z and w by $az + b$ and $\alpha w + \beta$, respectively, or by $(az + b)/(cz + d)$ and $(\alpha z + \beta)/(\gamma z + \delta)$ when the conics involved are circles, and interchanging z and w when necessary, we may reduce our conformal maps to the following types, arranged according to the number of parameters in the families of conics:

Five parameters (one type):

$$w = z; \tag{I}$$

any conic in the z-plane is transformed into the same conic in the w-plane.

Three parameters (two types):

$$w = 1/z; \tag{II}$$

any circle or straight line is transformed into a circle or straight line.

$$w = z^2; \tag{III}$$

denoting the parameters by k, r and θ, any ellipse $(k>1)$ or hyperbola $(1>k>-1)$

$$\bar{z} = k\frac{\bar{z}_1}{z_1}z + \frac{\bar{z}_1}{z_1}\sqrt{(k^2 - 1)(z^2 - z_1^2)}$$

with center at the origin and foci at

$$z_1 = re^{\theta i},\ z_2 = -z_1$$

is transformed into the ellipse $(k>1)$ or hyperbola $(1>k>-1)$

$$\bar{w} = K\frac{\bar{w}_1}{w_1}w + \frac{1 - K}{z}\bar{w}_1 + \frac{\bar{w}_1}{w_1}\sqrt{(K^2 - 1)w(w - w_1)}$$

with $K = 2k^2-1$ and foci at

$$w_1 = z_1^2 = r^2e^{2\theta i},\ w_2 = 0.$$

In the special case $k = 0$, the equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin go into the straight lines

$$e^{-2\theta i}w + e^{2\theta i}\bar{w} = r^2$$

Two parameters, denoted by r, θ or p, θ (four types):

$$w^2 = z^2 - 1; \qquad \text{(IV)}$$

any equilateral hyperbola

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = r^2$$

with center at the origin and foci at $z_1 = re^{\theta i}$, $z_2 = -z_1$ is transformed into the equilateral hyperbola

$$e^{-2\theta i}w^2 + e^{2\theta i}\bar{w}^2 = r^2 - 2\cos 2\theta$$

with center at the origin and foci at

$$w_1 = \sqrt{r^2 - 2\cos 2\theta}\; e^{\theta i},\ w_2 = -w_1.$$

In the special case $r^2 = 2\cos 2\theta$, we have a pair of perpendicular straight lines through the origin in the w-plane.

$$w = z^2; \qquad \text{(V)}$$

any straight line

$$e^{1/_2\theta i}\bar{z} - e^{-1/_2\theta i}z = i\sqrt{p}$$

is transformed into the parabola

$$e^{1/_2\theta i}\sqrt{\bar{w}} - e\sqrt{\;}^{1/_2\theta i}w = i\sqrt{p}$$

with focus at the origin and at a distance of $^1/_2p$ from the directrix, θ being the angle between the axis of the parabola and the real axis in the w-plane.

$$w = (z^2 - c_1)^2,\ \text{where } c_1 = 0 \text{ or } 1; \qquad \text{(VI)}$$

any equilateral hyperbola

$$e^{1/_2\theta i}\bar{z}^2 - e^{-1/_2\theta i}z^2 = i(\sqrt{p} + 2c_1 \sin {}^1/_2\theta)$$

with center at the origin and foci at

$$z_1 = e^{1/_4(\theta - \pi)i}(\sqrt{p} + 2c_1 \sin {}^1/_2\theta)^{1/_2},\ z_2 = -z_1$$

is transformed into the parabola

$$e^{1/_2\theta i}\sqrt{\bar{w}} - e^{-1/_2\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

$$w = (\sqrt{z} - 1)^2; \qquad \text{(VII)}$$

any parabola

$$e^{1/_2\theta i}\sqrt{\bar{z}} - e^{-1/_2\theta i}\sqrt{z} = i(\sqrt{p} + 2\sin {}^1/_2\theta)$$

with focus at the origin is transformed into the parabola

$$e^{1/_2\theta i}\sqrt{\bar{w}} - e^{-1/_2\theta i}\sqrt{w} = i\sqrt{p}$$

with focus at the origin.

One parameter, denoted by r, θ or p (nine principal types, the linear composition of each of these with itself or with another giving a total of forty-five types). There are tabulated below the nine functions which map the one parameter families of conics specified on the parallels to the real axis in the Z-plane. When $\varphi_1(z)$ and $\varphi_2(z)$ are any two of these functions (identical or different), all maps transforming one parameter families of conics into others are found by making

$$\varphi_2(w) = A\varphi_1(z) + B,$$

where A and B are constants and A real, and then making the linear transformations on z and w indicated at the beginning of this summary.

$$Z = z; \qquad \text{(VIII)}$$

all parallels to the real axis (special case of I).

$$Z = \log z; \qquad \text{(IX)}$$

all straight lines through $z = 0$ (the circles with center at $z = 0$ go into parallels to the imaginary Z-axis).

$$Z = i \log z; \qquad \text{(X)}$$

all circles with center at $z = 0$ (the straight lines through $z = 0$ go into parallels to the imaginary Z-axis).

$$Z = z^2; \qquad \text{(XI)}$$

all equilateral hyperbolas $z^2 + \bar{z}^2 = r^2$ with center at the origin and foci on the real axis (special case of III).

$$Z = \log (z^2-1); \qquad \text{(XII)}$$

all equilateral hyperbolas

$$e^{-2\theta i}z^2 + e^{2\theta i}\bar{z}^2 = 2 \cos 2\theta$$

with center at the origin and foci $\pm \sqrt{2 \cos 2\theta}\, e^{\theta i}$ (which are the end-points of the diameter with angle of slope θ of the lemniscate with foci at $+1$ and -1). These hyperbolas all pass through the foci of the lemniscate.

$$Z = \log (z + \sqrt{z^2-1}) \text{ or } z = {}^1\!/_2(e^Z + e^{-Z}); \qquad \text{(XIII)}$$

all hyperbolas with foci at $+1$ and -1 (all ellipses with these foci go into parallels to the imaginary Z-axis).

$$Z = 1/i \log (z + \sqrt{z^2-1}) \text{ or } z = \cos Z; \qquad \text{(XIV)}$$

all ellipses with foci at $+1$ and -1 (all hyperbolas with these foci go into the parallels to the imaginary Z-axis).

$$Z = \sqrt{z}; \qquad \text{(XV)}$$

all parabolas $\sqrt{\bar{z}}-\sqrt{z} = i\sqrt{p}$ with focus at the origin and the real axis as axis (special case of V).

$$Z = \log (\sqrt{z}-1); \qquad \text{(XVI)}$$

all parabolas

$$e^{1/_2\theta i}\sqrt{\bar{z}} - e^{-1/_2\theta i}\sqrt{z} = 2i \sin {}^1\!/_2\theta$$

with focus at the origin and passing through the point $z = 1$.

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[1] Von der Mühll, *J. Math., Berlin,* **69,** 1868 (264).

[2] Meyer, "Ueber die . . . Isothermen," *Dissertation,* Göttingen, 1879.